A Universal Crease Pattern
for Folding Orthogonal Shapes

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Abstract

We present a universal crease pattern—known in geometry as the *tetrakis tiling* and in origami as *box pleating*—that can fold into any object made up of unit cubes joined face-to-face (*polycubes*). More precisely, there is one universal finite crease pattern for each number n of unit cubes that need to be folded. This result contrasts previous universality results for origami, which require a different crease pattern for each target object, and confirms intuition in the origami community that box pleating is a powerful design technique.

1   Introduction

An early result in computational origami is that every polyhedral surface can be folded from a large enough square of paper [DDM00]. But each such folding uses a different crease pattern. Into how many different shapes can a single crease pattern fold?

Our motivation is developing programmable matter out of a foldable sheet. The idea is to statically manufacture a sheet with specific creases, and then dynamically program how much to fold each crease in the sheet. Thus a single manufactured sheet can be programmed to fold into anything that the single crease pattern (or a subset thereof) can fold.

We prove a universality result: a single $n \times n$ crease pattern can fold into all face-to-face gluings of $O(n)$ cubes. Thus, by setting the resolution n sufficiently large, we can fold any 3D solid up to a desired accuracy.
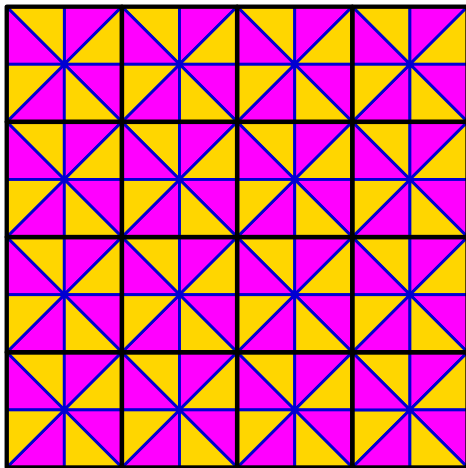
Figure 1: A 4 × 4 region of the tetrakis tiling.

Our crease patterns are finite (rectangular) portions of a single infinite tiling. The *tetrakis tiling* [GS87] is formed from the unit square grid by subdividing each square in half vertically, horizontally, and by the two diagonals, forming eight right isosceles triangles; see Figure 1. Note the scaling: one unit is the side length of an original square. Equivalently, the tetrakis tiling can be formed from a half-unit square grid with squares filled alternately with positive- and negative-slope diagonals.

A *tetrakis crease pattern* is a rectangular region (with integer coordinates) of the tetrakis tiling. Tetrakis crease patterns are similar to a style of origami called *box pleating* in which all creases are horizontal, vertical, or diagonal, but their endpoints may not lie on the integer grid. Box pleating was developed and popularized by Neal Elias in the 1960s [Kir98], and explored more mathematically in [Lan03]. Our universality result may have been suspected by origami artists, but has not been proved until now.

2   Definitions

We start with a few definitions about origami, specified somewhat informally for brevity. For more formal definitions, see [DO07, ch. 11].

For our purposes, a *piece of paper* is a two-dimensional surface (formally, a metric 2-manifold). A *crease pattern* is graph drawn on the piece of paper with straight edges and no crossings; edges are called *creases*. An *angle assignment* is an assignment of real numbers in [−180°, +180°] to creases in

the crease pattern, specifying a fold angle (negative for valley, positive for mountain). We allow a crease to be assigned an angle of 0, in which case we call the crease *trivial*, though we do not draw trivial creases in figures. A crease pattern and angle assignment determine the 3D geometry of a folded state, where each face maps as a 3D polygon via a rigid motion (isometry). A *folded state* consists of this geometry together with an *overlap order* defining the stacking relationship among faces of the crease pattern that touch in the 3D geometry, allowing the paper to touch but not cross itself. We will specify such overlap orders visually using diagrams that exaggerate the infinitesimal space between layers.

A *polycube* is an interior-connected union of unit cubes from the unit cube lattice. The *dual graph* of a polycube has a vertex for each unit cube and an edge between two vertices whose corresponding cubes share a face. By the interior-connected property, the dual graph is connected. The *faces* of the polycube are the (square) faces of the individual cubes that are not shared by any other cubes.

A *folding of a polycube* is a folded state that covers all faces of the polycube, and nothing outside the polycube. In particular, we allow the folded state to cover squares of the cubic lattice interior to the polycube. (Indeed, our foldings cover all such squares.) A face of the folded polycube is *seamless* if the outermost layer of paper covering it is an uncreased unit square of paper.

3   Folding Polycubes

In this section, we describe an algorithm for folding a given n-cube polycube from a square sheet of paper with crease pattern equal to an $O(n) \times O(n)$ region of the tetrakis tiling.

Theorem 1 *Any polycube of n cubes can be folded from a tetrakis crease pattern on a $(4n + 1) \times (2n + 1)$ rectangle of paper, with all faces seamless and made from one side of the paper, except for one specified face which has seams.*

Proof: The base case is $n = 1$. Figures 2 and 3 show the crease pattern and folded state, respectively, for a single cube folded from a 5×3 rectangular sheet of paper, which is within the desired bound. For this base case, we fold just the shaded 5×3 part of the crease pattern in Figure 2, making exactly the desired cube. Although hidden in Figure 3, the bottom face of the cube is indeed covered, with seams. All other faces are seamless.
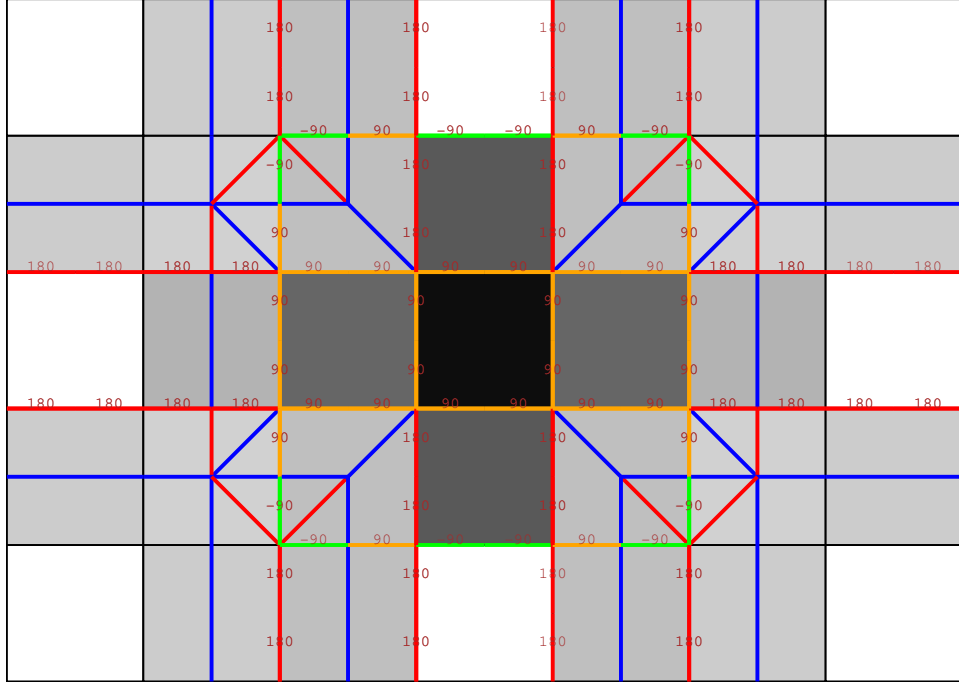
Figure 2: Crease pattern for a folding a single unit cube. Red line segments are mountain folds by 180°, orange segments are 90° mountain folds, green segments are 90° valley folds, and blue segments are 180° valley folds.
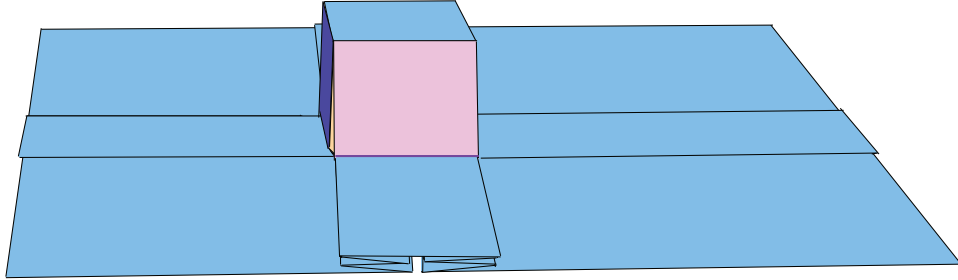

Figure 3: Folding of a unit cube from the crease pattern in Figure 2.

It remains to prove the inductive step.

Consider a polycube P of n cubes and a specified face $g \in P$ for seams. Let b be the unique cube having g as a face. Let T be a spanning tree of the dual graph of P, Because every tree has at least two leaves, T has a leaf corresponding to a cube $l \neq b$. Let t be the unique cube sharing a face

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with l, and let f be the face shared by t and l.

Now consider the polycube $P' = P \setminus \{l\}$, with $n-1$ cubes. Because $l \neq b$, g remains a face of P'. By induction, the $(4(n-1)+1) \times (2(n-1)+1)$ tetrakis crease pattern folds C' into P' with an angle assignment A', without seams on all faces but g. By symmetry we can assume that all faces are made from the top side of the paper.

We modify (C', A') as follows to obtain an angle assignment A for the $(4n+1) \times (2n+1)$ tetrakis crease pattern C folding into P, without seams on all faces but g. Because f is a face of P' and $f \neq g$, the folding of (C', A') has f seamless. Hence there is a unique unit square s of C' corresponding to the outermost layer of paper covering f. Suppose that s lies in row i and column j of C'. We insert two columns to the left of column j, two columns to the right of column j, a single row above row i, and a single row below row i, and add the crease pattern shown in Figure 4. In particular, we have replaced s by the 5×3 crease pattern for a unit cube, and we filled the remainder of the added rows and columns with the creases shown in Figure 2, We also reflect the creases in the original row i into the added rows, and similarly for the creases in the original column j, as described below. The new crease pattern C' has width $(4(n-1)+1)+4 = 4n+1$ and height $(2(n-1)+1)+2 = 2n+1$ as desired.
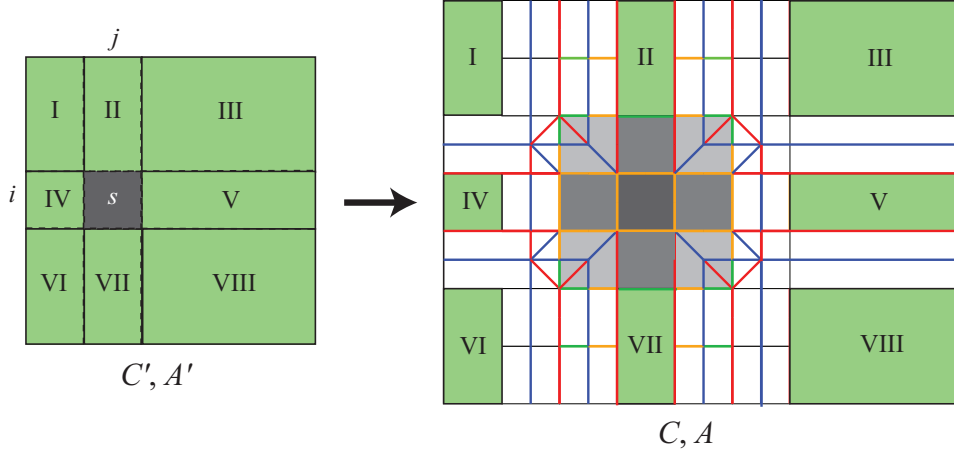

Figure 4: Modifying (C', A') (green) to produce (C, A). Crease coloring is the same as Figure 2.

Finally we show that the constructed crease pattern C and angle assignment A fold into the desired polycube P. We construct the folded state in two steps. First, we fold using just the inserted crease pattern, producing a

sheet with a cube l sticking out in place of s, as in Figure 3. As mentioned in the base case, all faces of l except f are seamless in this folding. Furthermore, all other unit squares of the sheet are seamless on the top side. Second, we apply the folding of (C', A') to this folded object, pretending that the cube l was just the unit square s. Because (C', A') does not fold s, the folding of (C', P') still works. The only difference is that all folds in row i and column j apply now to three layers, not just one, causing additional creases in the inserted rows and columns. Because all faces of P' are made from the top side of the paper, the folding remains seamless on all faces of P' except f and g. Face f is now the cube l, which means that we have folded P. □

A simple extension makes the folding entirely seamless:

Corollary 2 *Any polycube of n cubes can be folded from a tetrakis crease pattern on a $(4n + 1) \times (2n + 2)$ rectangle of paper, with all faces seamless and made from one side of the paper.*

Proof: Let P be a polycube of n cubes and let f be any of its faces. We compute the folding from Theorem 1 of a $(4n+1) \times (2n+1)$ rectangle into P, seamless except for f. We add an extra column on the right, extending any nontrivial horizontal creases into this column. The resulting crease pattern and angle assignment fold into the desired polycube, with an extra seamless square attached along an edge of f. We fold the seamless square on top of f to obtain an entirely seamless folding. □

Finally we show that a slightly more careful construction improves the size of the required square of paper.

Theorem 3 *Any polycube of n cubes can be folded from a tetrakis crease pattern on a square of paper of side length $3n + 2$, with all faces seamless and made from one side of the paper.*

Proof: We follow the same construction as Theorem 1, but through the induction on n, we alternate between the same modification and the $90°$ rotation of the modification. In other words, in odd steps we add four rows and two columns, and in even steps we add two rows and four columns. Thus we add three rows and columns on average per step, starting from a 5×3 rectangle. For n odd, we have an additional row, which is accounted for by the $+2$ (instead of $+1$). In all cases, we have an additional column of paper, and for n even, we have an additional row as well. Folding these over in sequence, similar to Corollary 2, removes the seams from the last face. □

Note that these bounds are tight up to constant factors for square paper, as folding an $n \times 1 \times 1$ tower of unit cubes requires starting from a square of side length $\Omega(n)$ in order to have diameter $\Omega(n)$, because the diameter of the tower is n and folding can only decrease diameter.

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